

Randomized Bicriteria Approximation Algorithm for Minimum Submodular Cost Partial Multi-Cover Problem

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Abstract

This paper studies randomized approximation algorithm for a variant of the set cover problem called *minimum submodular cost partial multi-cover* (SCPMC).

In a *partial set cover problem*, the goal is to find a minimum cost sub-collection of sets covering at least a required fraction of elements. In a *multi-cover problem*, each element e has a covering requirement r_e , and the goal is to find a minimum cost sub-collection of sets \mathcal{S}' which fully covers all elements, where an element e is fully covered by \mathcal{S}' if e belongs to at least r_e sets of \mathcal{S}' . In a *minimum submodular cost set cover problem* (SCSC), the cost function on sub-collection of sets is submodular and the goal is to find a set cover with the minimum cost.

The SCPMC problem studied in this paper is a combination of the above three problems, in which the cost function on sub-collection of sets is submodular and the goal is to find a minimum cost sub-collection of sets which fully covers at least q -percentage of all elements. Previous work shows that such a combination enormously increases the difficulty of studies, even when the cost function is linear.

In this paper, assuming that the maximum covering requirement $r_{\max} = \max_e r_e$ is a constant and the cost function is nonnegative, monotone nondecreasing, and submodular, we give the first randomized bicriteria algorithm for SCPMC the output of which fully covers at least $(q - \varepsilon)$ -percentage of all elements and the performance ratio is $O(b/\varepsilon)$ with a high probability, where $b = \max_e \binom{f}{r_e}$ and f is the maximum number of sets containing a common element. The algorithm is based on a novel non-linear program. Furthermore, in the case when the

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covering requirement $r \equiv 1$, a bicriteria $O(f/\varepsilon)$ -approximation can be achieved even when monotonicity requirement is dropped off from the cost function.

Keywords: partial cover, multi-cover, submodular cover, Lovász extension, randomized algorithm, approximation algorithm, bicriteria.

1 Introduction

Set Cover is one of the most important combinatorial optimization problems in both the theoretical field and the application field, the goal of which is to find a sub-collection of sets with the minimum cost to cover all elements. There are a lot of variants of the set cover problem. The minimum *partial set cover* problem (PSC) is to find a minimum cost sub-collection of sets to cover at least q -percentage of all elements. One motivation of PSC comes from the phenomenon that in a real world, “satisfying all requirements” will be too costly or even impossible, because of resource limitation or political policy. Another variant is the minimum *multi-cover* problem (MC), which comes from the requirement of fault tolerance in practice. In MC, each element e has a covering requirement r_e , and the goal is to find a minimum cost sub-collection \mathcal{S}' to fully cover all elements, where element e is fully covered by \mathcal{S}' if e belongs to at least r_e sets of \mathcal{S}' . Another generalization of set cover is *submodular cost set cover* (SCSC), in which the cost function on sub-collection of sets is submodular and the goal is to find a set cover with the minimum cost. Submodular functions have a natural diminishing returns property which finds wide applications in the real world, including economics, game theory, machine learning and computer vision, etc.

In this paper, we consider a problem which is a combination of the above three problems. In the *minimum submodular cost partial multi-cover* problem (SCPMC), each element has a profit as well as a covering requirement, the goal is to find a minimum submodular cost sub-collection of sets such that the profit of fully covered elements is at least a fixed percentage of the total profit.

1.1 Related Work

For Set Cover, Hochbaum [9] gave an f -approximation algorithm based on LP rounding where f is the maximum number of sets containing a common element. Khot and Regev [13] showed that the set cover problem cannot be approximated within $f - \varepsilon$ for any constant $\varepsilon > 0$ assuming that unique games conjecture is true. Another classic result on Set Cover is that greedy strategy yields a $\ln \Delta$ -approximation [5, 11, 17], where Δ is the maximum cardinality of a set. Dinur and Steurer [4] showed that the set cover problem cannot be approximated to $(1 - o(1)) \ln n$ unless $P = NP$, where n is the size of ground set.

For MC, Dobson [6] gave an H_K -approximation algorithm for the *minimum multi-set multi-cover problem* (MSMC), where K is the maximum size of a multi-set and $H_K = \sum_{i=1}^K 1/i$ is the harmonic number (recall that $H_K \approx \ln K$). Rajagopalan and

Vazirani [18] gave a greedy algorithm achieving the same performance ratio, using dual fitting analysis. For the *minimum set k -cover problem* in which the covering requirement of every element is k , Berman *et al.* [2] gave a randomized algorithm achieving expected performance ratio at most $\ln(\frac{\Delta}{k})$.

For PSC, Kearns [12] gave the first greedy algorithm achieving performance ratio $(2H_n + 3)$. Refining the greedy algorithm, Slavik [21] improved the ratio to $H_{\min\{[qn], \Delta\}}$, where q is the desired covering ratio. Using primal dual method, Gandhi *et al.* [8] obtained an f -approximation. Bar-Yehuda [1] studied a generalized version of the partial cover problem in which each element has a profit. Using local ratio method, he also obtained an f -approximation. Proposing an Lagrangian relaxation framework, Konemann *et al.* [14] gave a $(\frac{4}{3} + \varepsilon)H_\Delta$ -approximation for the generalized partial cover problem.

From the above related work, it can be seen that both PSC and MC admit performance ratios which match those best ratios for the classic set cover problem. However, combining partial cover with multi-cover seems to enormously increase the difficulty of studies. Ran *et al.* [19] were the first to study approximation algorithm for the *minimum partial multi-cover problem* (PMC). Using greedy strategy and a delicate dual fitting analysis, they gave a γH_Δ -approximation algorithm, where $\gamma = 1/(1 - (1 - q)\eta)$, $\eta = \Delta \frac{c_{\max}}{c_{\min}} \frac{r_{\max}}{r_{\min}}$, and c_{\max} , c_{\min} are the maximum and the minimum cost of set, r_{\max} , r_{\min} are the maximum and the minimum covering requirement of element, respectively. This ratio is meaningful only when the covering percentage q is very close to 1. In [20], Ran *et al.* presented a simple greedy algorithm achieving performance ratio Δ . Recall that in terms of Δ , greedy algorithm for Set Cover achieves performance ratio $\ln \Delta$. So, ratio Δ for PMC is exponentially larger than the one for Set Cover. In the same paper, they also presented a local ratio algorithm which reveals an interesting “shock wave” phenomenon: their performance ratio is f for both PSC (that is, when $r_{\max} = r_{\min} = 1$ which is the partial *single* cover problem) and MC (that is, when $q = 1$ which is the *full* multi-cover problem); however, when q is smaller than 1 by a very small constant, the ratio jumps abruptly to $O(n)$.

The submodular cost set cover problem was first proposed by Iwata and Nagano [10]. They gave an f -approximation algorithm for nonnegative submodular functions. In paper [15], Koufogiannakis and Young generalized set cover constraint to arbitrary covering constraints and gave an f -approximation algorithm for monotone nondecreasing nonnegative submodular functions.

In this paper we combine submodular cost function with partial multi-cover constraint. As one can see from previous results on PMC, even when the cost function is linear, the partial multi-cover problem is already very difficult.

1.2 Our Contribution

The major contribution of this paper is a randomized $(\varepsilon, O(\frac{b}{\varepsilon}))$ -approximation algorithm for SCPMC, that is, the algorithm produces a solution covering at least $(q - \varepsilon)$ -percentage of the total covering requirement, and achieves performance ratio $O(\frac{b}{\varepsilon})$

with a high probability, where $b = \max_e \binom{f}{r_e}$, and f is the maximum number of sets containing a common element.

Before presenting this algorithm, we show that a natural integer program for SCPMC does not work since its integrality gap is arbitrarily large. Hence, to obtain a good approximation, we propose a novel integer program. The relaxation of the integer program uses Lovász extension [16]. Our algorithm consists of two stages of rounding. The first stage is a deterministic rounding. The second stage is a random rounding, the analysis of which is based on an equivalent expression of Lovász extension [3] in view of expectation.

As far as we know, this is the first approximation algorithm for a partial version of the submodular multi-cover problem. Furthermore, we show that for the special case when the covering requirement $r \equiv 1$ (the special case is abbreviated as SCPSC), our method can be adapted to yield an $(\varepsilon, O(f/\varepsilon))$ -approximation with high probability, even when monotonicity is dropped off from the requirement of the cost function.

This paper is organized as follows. In Section 2, we introduce formal definitions of problems considered in this paper, as well as some technical results. The bicriteria randomized algorithm for SCPMC is presented and analyzed in Section 3. In Section 4, we show how to adapt our algorithm to deal with SCPSC. The last section concludes the paper and discusses some future work.

2 Preliminaries

Definition 2.1 (Submodular Cost Partial Multi-Cover (SCPMC)). Suppose E is an element set and $\mathcal{S} \subseteq 2^E$ is a collection of subsets of E with $\bigcup_{S \in \mathcal{S}} S = E$; each element $e \in E$ has a positive covering requirement r_e and a positive profit p_e ; cost function $\rho_0 : 2^{\mathcal{S}} \mapsto \mathbb{R}$ is defined on sub-collections of \mathcal{S} , which is nonnegative, monotone nondecreasing, and submodular. Given a constant $q \in (0, 1]$ called *covering ratio*, the SCPMC problem is to find a minimum cost sub-collection \mathcal{S}' such that $\sum_{e \sim \mathcal{S}'} p(e) \geq qP$, where $P = \sum_{e \in E} p(e)$ is the total profit, $e \sim \mathcal{S}'$ means that e is *fully covered* by \mathcal{S}' , that is, $|\{S \in \mathcal{S}' : e \in S\}| \geq r_e$. An instance of SCPMC is denoted as $(E, \mathcal{S}, r, p, q, \rho_0)$.

In particular, when $r_{\max} = 1$, we call the problem a *submodular cost partial set cover problem* (SCPSC). When the cost function is linear, that is, every set $S \in \mathcal{S}$ has a cost $c(S)$ and the cost of a sub-collection \mathcal{S}' is $\rho_0(\mathcal{S}') = \sum_{S \in \mathcal{S}'} c(S)$, the problem is exactly the minimum *partial multi-cover problem* (PMC).

Submodular function has many equivalent definitions. We only introduce the following one which is convenient to be used in this paper.

Definition 2.2 (submodular function). Given a ground set E , a set function $\rho : 2^E \mapsto \mathbb{R}$ is *submodular* if for any $E'' \subseteq E' \subseteq E$ and $E_0 \subseteq E \setminus E'$, we have

$$\rho(E' \cup E_0) - \rho(E') \leq \rho(E'' \cup E_0) - \rho(E''). \quad (1)$$

Notice that a nonnegative submodular function ρ satisfies *subadditivity*: for any sets $X, Y \subseteq E$,

$$\rho(X \cup Y) \leq \rho(X) + \rho(Y). \quad (2)$$

Notice that a set $S \subseteq E$ can be indicated by its characteristic vector $x_S = (x_1, \dots, x_n)$, where $n = |E|$, $E = \{e_1, \dots, e_n\}$, and $x_i = 1$ if $e_i \in S$ and $x_i = 0$ if $e_i \notin S$. So, in the following, we shall use notation $\{0, 1\}^n \mapsto \mathbb{R}$ to refer to a set function. The relationship between submodularity and convexity can be formulated in terms of Lovász extension.

Definition 2.3 (Lovász extension [16]). For a set function $\rho : \{0, 1\}^n \mapsto \mathbb{R}$, the Lovász extension $\hat{\rho} : \mathbb{R}^n \rightarrow \mathbb{R}$ is defined as follows. For any vector $x \in \mathbb{R}^n$, order elements as $e_{j_1}, e_{j_2}, \dots, e_{j_n}$ such that $x_{j_1} \geq x_{j_2} \geq \dots \geq x_{j_n}$, where x_{j_i} is the coordinate of x indexed by e_{j_i} . Let $E_i = \{e_{j_1}, e_{j_2}, \dots, e_{j_i}\}$. The value of $\hat{\rho}$ at x is

$$\hat{\rho}(x) = \sum_{i=1}^{n-1} (x_{j_i} - x_{j_{i+1}}) \rho(E_i) + x_{j_n} \rho(E_n). \quad (3)$$

The above definition implies that Lovász extension $\hat{\rho}$ satisfies *positive homogenous property*, that is, for any $t > 0$, $\hat{\rho}(tx) = t\hat{\rho}(x)$. The following result reveals the relationship between submodularity and convexity.

Theorem 2.4. *A set function ρ is submodular if and only if its Lovász extension $\hat{\rho}$ is convex.*

The following is an equivalent expression of Lovász extension in range $[0, 1]^n$.

Theorem 2.5 ([3]). Let ρ be a set function $\{0, 1\}^n \mapsto \mathbb{R}$. The Lovász extension $\hat{\rho}$ of ρ in range $[0, 1]^n$ can be equivalently expressed as

$$\hat{\rho}(x) = \mathbb{E}_{\theta \in [0, 1]} [\rho(x^\theta)] = \int_0^1 \rho(x^\theta) d\theta, \quad (4)$$

where $x_i^\theta = 1$ if $x_i \geq \theta$, otherwise $x_i^\theta = 0$.

In this paper, we study the SCPMC problem under the following assumptions.

(Assumption 1) The maximum covering requirement $r_{\max} = \max\{r_e : e \in E\}$ has a constant upper bound.

(Assumption 2) Since submodular cost (full) multi-cover problem is already studied in [10, 15], we only consider the partial version, that is, it is assumed that $q < 1$.

3 Approximation Algorithm for SCPMC

A natural idea to model the SCPMC problem is to use the following integer program:

$$\begin{aligned}
& \min \rho_0(x) \\
& s.t. \quad \sum_{e: e \in E} p_e y_e \geq qP, \\
& \quad \sum_{S: e \in S} x_S \geq r_e y_e, \text{ for any } e \in E \\
& \quad x_S \in \{0, 1\} \text{ for } S \in \mathcal{S} \\
& \quad y_e \in \{0, 1\} \text{ for } e \in E
\end{aligned} \tag{5}$$

Here x_S indicates whether set S is selected and y_e indicates whether element e is fully covered. The second constraint says that if $y_e = 1$ then at least r_e sets containing e must be selected and thus e is fully covered. Relaxing (5), we have the following convex program:

$$\begin{aligned}
& \min \hat{\rho}_0(x) \\
& s.t. \quad \sum_{e: e \in E} p_e y_e \geq qP, \\
& \quad \sum_{S: e \in S} x_S \geq r_e y_e, \text{ for any } e \in E \\
& \quad x_S \geq 0 \text{ for } S \in \mathcal{S} \\
& \quad 1 \geq y_e \geq 0 \text{ for } e \in E
\end{aligned} \tag{6}$$

However, based on such a program, one cannot find a good approximation. The following example shows that the integrality gap between (5) and (6) can be arbitrarily large, even when the profit function is a constant and the cost function is linear.

Example 3.1. Let $E = \{e_1, e_2\}$, $\mathcal{S} = \{S_1, S_2, S_3\}$ with $S_1 = \{e_1\}$, $S_2 = \{e_2\}$, $S_3 = \{e_1, e_2\}$, $c(S_1) = c(S_2) = 1$, $c(S_3) = M$ where M is a large positive number, $r(e_1) = r(e_2) = 2$, $p(e_1) = p(e_2) = 1$, $q = 1/2$, and the cost function $\rho_0(x) = \sum_{S \in \mathcal{S}} c(S)x_S$. Then $x_{S_1} = x_{S_2} = 1$, $x_{S_3} = 0$, $y_{e_1} = y_{e_2} = 1/2$ form a feasible solution to (6) with objective value 2, while any integral feasible solution to (5) has cost at least $M + 1$.

Hence, to obtain a good approximation, we need to find another program.

3.1 Integer Program and Convex Relaxation

For an element e , an r_e -cover is a sub-collection $\mathcal{A} \subseteq \mathcal{S}$ with $|\mathcal{A}| = r_e$ such that $e \in S$ for every $S \in \mathcal{A}$. Denote by Ω_e the family of all r_e -covers and $\Omega = \bigcup_{e \in E} \Omega_e$. The following example illustrates these concepts.

Example 3.2. Let $E = \{e_1, e_2, e_3\}$. $\mathcal{S} = \{S_1, S_2, S_3\}$ with $S_1 = \{e_1, e_2\}$, $S_2 = \{e_1, e_2, e_3\}$, $S_3 = \{e_2, e_3\}$, $S_4 = \{e_1, e_3\}$, and $r(e_1) = 2$, $r(e_2) = r(e_3) = 1$. For this example, $\Omega_{e_1} = \{\{S_1, S_2\}, \{S_1, S_4\}, \{S_2, S_4\}\}$, $\Omega_{e_2} = \{\{S_1\}, \{S_2\}, \{S_3\}\}$, $\Omega_{e_3} = \{\{S_2\}, \{S_3\}, \{S_4\}\}$, and $\Omega = \{\{S_1\}, \{S_2\}, \{S_3\}, \{S_4\}, \{S_1, S_2\}, \{S_1, S_4\}, \{S_2, S_4\}\}$.

Let $\rho: 2^\Omega \rightarrow \mathbb{R}$ be the function on sub-families of Ω defined by

$$\rho(\Omega') = \rho_0\left(\bigcup_{\mathcal{A} \in \Omega'} \mathcal{A}\right) \quad (7)$$

for $\Omega' \subseteq \Omega$. For example, $\rho(\{\{S_1\}, \{S_1, S_2\}\}) = \rho_0(\{S_1, S_2\})$. The SCPMC problem can be modeled as an integer program as follows.

$$\begin{aligned} & \min \rho(x) \\ & s.t. \quad \sum_{e: e \in E} p_e y_e \geq qP, \\ & \quad \sum_{\mathcal{A}: \mathcal{A} \in \Omega_e} x_{\mathcal{A}} \geq y_e, \text{ for any } e \in E \\ & \quad x_{\mathcal{A}} \in \{0, 1\} \text{ for } \mathcal{A} \in \Omega \\ & \quad y_e \in \{0, 1\} \text{ for } e \in E \end{aligned} \quad (8)$$

Here, $x_{\mathcal{A}}$ indicates whether cover \mathcal{A} is selected and y_e indicates whether element e is fully covered. The second constraint says that if $y_e = 1$, then at least one r_e -cover must be selected and thus e is fully covered.

Example 3.3. For the example in Example 3.2, suppose $p_{e_i} \equiv 1$ for $i = 1, 2, 3$ and $q = 2/3$. Consider a feasible solution to (8): $x_{\mathcal{A}_1} = x_{\mathcal{A}_2} = 1$ for $\mathcal{A}_1 = \{S_1, S_2\}$, $\mathcal{A}_2 = \{S_2\}$, and $x_{\mathcal{A}} = 0$ for all other $\mathcal{A} \in \Omega \setminus \{\mathcal{A}_1, \mathcal{A}_2\}$, we have $y_{e_1} = y_{e_2} = 1$ and $y_{e_3} = 0$. This feasible solution to (8) has objective value $\rho(\{\mathcal{A}_1, \mathcal{A}_2\}) = \rho_0(S_1, S_2)$, which corresponds to a feasible solution $\{S_1, S_2\}$ to SCPMC with the same cost. Conversely, for the feasible solution $\{S_1, S_2\}$ to SCPMC, it is natural to set $x_{\mathcal{A}_1} = 1$ and all other $x_{\mathcal{A}}$ to be zeros. However, this is not a feasible solution to (8). Nevertheless, one can construct a feasible solution to (8) having the same cost by setting $x_{\mathcal{A}_1} = x_{\mathcal{A}_2} = 1$ and all other $x_{\mathcal{A}}$ to be zeros.

In general, for a feasible solution \mathcal{S}' to SCPMC, one can construct a feasible solution to (8) as follows: for each element e which is fully covered by \mathcal{S}' , let $y_e = 1$ and let $x_{\mathcal{A}_e} = 1$ for exactly one r_e -cover \mathcal{A}_e which contains r_e subsets of \mathcal{S}' (such \mathcal{A}_e exists since e is fully covered by \mathcal{S}'); all other variables are set to be zeros. Such a construction clearly results in a feasible solution to (8) whose objective value is at most $\rho_0(\mathcal{S}')$ (by the monotonicity of ρ_0). So, (8) is indeed a characterization of the SCPMC problem.

The following lemma shows that function ρ is nonnegative, monotone nondecreasing, and submodular.

Lemma 3.4. *If ρ_0 is nonnegative, monotone nondecreasing, and submodular, then the function ρ defined in (7) is also nonnegative, monotone nondecreasing, and submodular.*

Proof. The nonnegativity and the monotonicity are obvious. To prove the submodularity, by Definition 2.2, it is sufficient to show that for any $\Omega'' \subseteq \Omega' \subseteq \Omega$ and $\Omega_0 \subseteq \Omega \setminus \Omega'$,

$$\rho(\Omega' \cup \Omega_0) - \rho(\Omega') \leq \rho(\Omega'' \cup \Omega_0) - \rho(\Omega''). \quad (9)$$

Denote $\bigcup_{\mathcal{A} \in \Omega'} \mathcal{A} = \mathcal{S}'$ and $\bigcup_{\mathcal{A} \in \Omega''} \mathcal{A} = \mathcal{S}''$. Since $\Omega'' \subseteq \Omega'$, we have $\mathcal{S}'' \subseteq \mathcal{S}'$. Denote $\mathcal{S}_1 = (\bigcup_{\mathcal{A} \in \Omega' \cup \Omega_0} \mathcal{A}) \setminus \mathcal{S}'$ and $\mathcal{S}_2 = (\bigcup_{\mathcal{A} \in \Omega'' \cup \Omega_0} \mathcal{A}) \setminus \mathcal{S}''$. Then $\mathcal{S}_1 \subseteq \mathcal{S}_2$. Combining this with the observation that $\mathcal{S}' \cup \mathcal{S}_1 = \bigcup_{\mathcal{A} \in \Omega' \cup \Omega_0} \mathcal{A} \supseteq \bigcup_{\mathcal{A} \in \Omega'' \cup \Omega_0} \mathcal{A} = \mathcal{S}'' \cup \mathcal{S}_2$, we have

$$\mathcal{S}'' \subseteq (\mathcal{S}'' \cup \mathcal{S}_2) \setminus \mathcal{S}_1 \subseteq \mathcal{S}'. \quad (10)$$

It follows that

$$\begin{aligned} \rho(\Omega' \cup \Omega_0) - \rho(\Omega') &= \rho_0(\mathcal{S}' \cup \mathcal{S}_1) - \rho_0(\mathcal{S}') \\ &\leq \rho_0(((\mathcal{S}'' \cup \mathcal{S}_2) \setminus \mathcal{S}_1) \cup \mathcal{S}_1) - \rho_0((\mathcal{S}'' \cup \mathcal{S}_2) \setminus \mathcal{S}_1) \\ &\leq \rho_0(((\mathcal{S}'' \cup \mathcal{S}_2) \setminus \mathcal{S}_1) \cup \mathcal{S}_1) - \rho_0(\mathcal{S}'') \\ &= \rho_0(\mathcal{S}'' \cup \mathcal{S}_2) - \rho_0(\mathcal{S}'') \\ &= \rho(\Omega'' \cup \Omega_0) - \rho(\Omega''), \end{aligned}$$

where the first inequality uses submodularity of ρ_0 and (10), and the second inequality uses the monotonicity of ρ_0 and (10). Inequality (9), and thus the lemma, is proved. \square

Remark 3.5. If ρ_0 is nonnegative and submodular but is not monotone nondecreasing, then ρ is not necessarily submodular. Consider the following example. Let $\mathcal{S} = \{S_1, S_2, S_3\}$ with $\rho_0(\{S_1\}) = \rho_0(\{S_1, S_3\}) = 1$ and $\rho_0(\mathcal{S}') = 0$ for any other sub-collection $\mathcal{S}' \subseteq \mathcal{S}$. It can be verified that ρ_0 is nonnegative and submodular. Consider sub-families $\Omega'' = \{\{S_1\}\} \subseteq \Omega' = \{\{S_1\}, \{S_1, S_2\}\}$ and $\Omega_0 = \{\{S_1, S_2, S_3\}\}$, it can be calculated that

$$\rho(\Omega' \cup \Omega_0) - \rho(\Omega') = 0 - 0 = 0 > -1 = 0 - 1 = \rho(\Omega'' \cup \Omega_0) - \rho(\Omega'').$$

So, ρ is not submodular.

Let $\hat{\rho}$ be the Lovász extension of ρ . By Theorem 2.4, $\hat{\rho}$ is convex. Relaxing (8), we have the following convex program:

$$\begin{aligned} &\min \hat{\rho}(x) \\ \text{s.t. } &\sum_{e: e \in E} p_e y_e \geq qP, \\ &\sum_{\mathcal{A}: \mathcal{A} \in \Omega_e} x_{\mathcal{A}} \geq y_e, \text{ for any } e \in E \\ &x_{\mathcal{A}} \geq 0 \text{ for } \mathcal{A} \in \Omega \\ &1 \geq y_e \geq 0 \text{ for } e \in E \end{aligned} \quad (11)$$

Lemma 3.6. *Convex program (11) is polynomial-time solvable.*

Proof. It is known that (see [7]) for a submodular function ρ , its Lov'asz extension $\hat{\rho}(x) = \rho^-(x)$ for any $x \in [0, 1]^{|\Omega|}$, where ρ^- is the *convex closure* of ρ defined as follows. For each sub-family Ω' of Ω , denote by $\chi_{\Omega'}$ as the indicator vector of Ω' . The convex closure of ρ is the function $\rho^-: [0, 1]^{|\Omega|} \mapsto \mathbb{R}$ such that for any vector $x \in [0, 1]^{|\Omega|}$, $\rho^-(x) = \min\{\sum_{\Omega' \subseteq \Omega} \lambda_{\Omega'} \rho(\Omega') : \sum_{\Omega' \subseteq \Omega} \lambda_{\Omega'} \chi_{\Omega'} = x, \sum_{\Omega' \subseteq \Omega} \lambda_{\Omega'} = 1, \lambda_{\Omega'} \geq 0\}$. Hence (11) can be rewritten as:

$$\begin{aligned}
& \min \sum_{\Omega' \subseteq \Omega} \lambda_{\Omega'} \rho(\Omega') \\
& s.t. \quad \sum_{\Omega' \subseteq \Omega} \lambda_{\Omega'} = 1, \\
& \quad \sum_{e: e \in E} p_e y_e \geq qP, \\
& \sum_{\Omega': \mathcal{A} \in \Omega' \subseteq \Omega} \lambda_{\Omega'} = x_{\mathcal{A}}, \text{ for any } \mathcal{A} \in \Omega \\
& \sum_{\mathcal{A}: \mathcal{A} \in \Omega_e} x_{\mathcal{A}} \geq y_e, \text{ for any } e \in E \\
& \lambda_{\Omega'} \geq 0 \text{ for } \Omega' \subseteq \Omega \\
& x_{\mathcal{A}} \geq 0 \text{ for } \mathcal{A} \in \Omega \\
& 1 \geq y_e \geq 0 \text{ for } e \in E
\end{aligned} \tag{12}$$

Notice that this is a linear program. For each element e , $|\Omega_e| \leq b = \max_e \binom{f}{r_e}$. Since in Assumption 1, we have assumed that r_{\max} is upper bounded by a constant, the number of variables in the form of $x_{\mathcal{A}}$ or y_e is polynomial. However, the number of variables in the form of $\lambda_{\Omega'}$ is exponential.

Consider the dual program of (12):

$$\begin{aligned}
& \max \quad a + bqP - \sum_{e \in E} f_e \\
& s.t. \quad a + \sum_{\mathcal{A} \in \Omega'} c_{\mathcal{A}} \leq \rho(\Omega'), \text{ for any } \Omega' \subseteq \Omega \\
& \quad \sum_{e: e \in \mathcal{A}} d_e - c_{\mathcal{A}} \leq 0, \text{ for any } \mathcal{A} \in \Omega \\
& \quad p_e b - d_e - f_e \leq 0, \text{ for any } e \in E \\
& \quad b \geq 0 \text{ and } d_e, f_e \geq 0 \text{ for } e \in E
\end{aligned} \tag{13}$$

Since both $|\Omega|$ and $|E|$ are polynomial, to solve (13), it suffices to construct a separation oracle for the first set of constraints.

Define $g(\Omega') = \rho(\Omega') - \sum_{\mathcal{A} \in \Omega'} c_{\mathcal{A}}$ for any $\Omega' \subseteq \Omega$. Since g is obtained by subtracting a modular function from a submodular function, g is also a submodular function. Hence, by finding a minimizer of g , which can be done in polynomial time, and then check whether its g -value is at least a , we can either claim the validity of the first set of constraints or find out a violated constraint. \square

Since (11) is a relaxation of (8), we have $opt_{cp} \leq opt$, where opt_{cp} is the optimal value of (11) and opt is the optimal integer value of (8) (which is also the optimal value of SCPMC).

3.2 Rounding Algorithm

For a sub-collection $\mathcal{S}' \subseteq \mathcal{S}$, denote by $\mathcal{C}(\mathcal{S}')$ the set of elements fully covered by \mathcal{S}' . Two parameters s, t are needed which are chosen in Theorem 3.11 to guarantee the desired ratio with high probability. The rounding algorithm consists of two phases. In the first phase, a deterministic rounding is executed to form a sub-collection \mathcal{S}_1 . In the second phase, a randomized rounding is executed to form a sub-collection \mathcal{S}_2 . The output is the union of \mathcal{S}_1 and \mathcal{S}_2 .

Algorithm 1 Algorithm for SCPMC

Input: A SCPMC instance $(E, \mathcal{S}, r, p, q, \rho_0)$, two parameters s, t satisfying $1 < t < s \leq 1/q$, and a real positive number $\varepsilon < q$.

Output: A sub-collection \mathcal{S}' which has total covering profit at least $(q - \varepsilon)P$.

- 1: Find an optimal solution (x^*, y^*) to (11).
 - 2: $\mathcal{S}_1 \leftarrow \emptyset, \mathcal{S}_2 \leftarrow \emptyset$.
 - 3: **for** all e with $y_e^* \geq \frac{1}{s}$ **do**
 - 4: For each $\mathcal{A} \in \Omega_e$ with $x_{\mathcal{A}}^* \geq \frac{1}{bs}$, let $\hat{x}_{\mathcal{A}} \leftarrow 1$.
 - 5: **end for**
 - 6: For all $x_{\mathcal{A}}^*$ which is not rounded up to 1, set $\hat{x}_{\mathcal{A}} \leftarrow 0$.
 - 7: $\mathcal{S}_1 \leftarrow \{S : S \in \mathcal{A} \text{ with } \hat{x}_{\mathcal{A}} = 1\}$.
 - 8: If \mathcal{S}_1 has total covering profit at least $(q - \varepsilon)P$ then output $\mathcal{S}' \leftarrow \mathcal{S}_1$ and stop.
 - 9: $E' \leftarrow E - \mathcal{C}(\mathcal{S}_1)$, $q' \leftarrow (qP - p(\mathcal{C}(\mathcal{S}_1)))/P$.
 - 10: **for** $i = 1$ to $s \ln(\frac{s}{s-t})b$ **do**
 - 11: Pick $\theta \in [0, 1]$ randomly uniformly.
 - 12: For each remaining \mathcal{A} with $x_{\mathcal{A}}^* \geq \theta$, set $\hat{x}_{\mathcal{A}} \leftarrow 1$ and $\mathcal{S}_2 \leftarrow \mathcal{S}_2 \cup \{S : S \in \mathcal{A}\}$.
 - 13: **end for**
 - 14: Output $\mathcal{S}' = \mathcal{S}_1 \cup \mathcal{S}_2$.
-

3.3 Approximation Analysis

Lemma 3.7. *For the collection of sets \mathcal{S}_1 computed by Algorithm 1, $\rho_0(\mathcal{S}_1) \leq bs \cdot opt_{cp}$. Furthermore, all elements with $y_e^* \geq \frac{1}{s}$ are fully covered by \mathcal{S}_1 .*

Proof. Let \hat{x} be the vector defined after Line 6 of Algorithm 1, and let z be the vector with $z_{\mathcal{A}} = \min\{1, bsx_{\mathcal{A}}^*\}$ for $\mathcal{A} \in \Omega$.

Recall that Lovász extension in Definition 2.3 requires an ordering of elements in a non-increasing manner. By the definition of z and by the nonnegativity of ρ , we can take the ordering of elements defining $\hat{\rho}(z)$ and $\hat{\rho}(bsx^*)$ to be the same and

$$\hat{\rho}(z) \leq \hat{\rho}(bsx^*). \quad (14)$$

We claim that $\hat{x}_{\mathcal{A}} \leq z_{\mathcal{A}}$ holds for any index $\mathcal{A} \in \Omega$. This is clearly true if $\hat{x}_{\mathcal{A}} = 0$. For an index \mathcal{A} with $\hat{x}_{\mathcal{A}} = 1$, we have $x_{\mathcal{A}}^* \geq 1/bs$ (by Line 4 of Algorithm 1), which implies $z_{\mathcal{A}} = 1$. The claim is proved. It follows that for any $\theta \in [0, 1]$ and for any index $\mathcal{A} \in \Omega$, $\hat{x}_{\mathcal{A}}^\theta \leq z_{\mathcal{A}}^\theta$ (recall the notation x_i^θ defined in Theorem 2.5). Then, by the monotonicity of ρ , we have

$$\rho(\hat{x}^\theta) \leq \rho(z^\theta). \quad (15)$$

Combining (14), (15) with the positive homogeneous property of Lovász extension,

$$\rho_0(\mathcal{S}_1) = \hat{\rho}(\hat{x}) = \int_0^1 \rho(\hat{x}^\theta) d\theta \leq \int_0^1 \rho(z^\theta) d\theta = \hat{\rho}(z) \leq \hat{\rho}(bsx^*) = bs\hat{\rho}(x^*) = bs \cdot \text{opt}_{cp}.$$

Next, consider the second half of the lemma. For each element e with $y_e^* \geq \frac{1}{s}$, by the second constraint of (11), and by the observation that $|\Omega_e| \leq b$, we have

$$\max_{\mathcal{A} \in \Omega_e} x_{\mathcal{A}}^* \geq y_e^*/b \geq 1/bs. \quad (16)$$

Hence there is at least one r_e -cover $\mathcal{A} \in \Omega_e$ with value $x_{\mathcal{A}}^* \geq \frac{1}{bs}$, and thus $\hat{x}_{\mathcal{A}} = 1$. That is, after the deterministic rounding, at least one r_e -cover is chosen into \mathcal{S}_1 , and thus e is fully covered. \square

Lemma 3.8. *For the collection of sets \mathcal{S}_2 computed by Algorithm 1, the expected cost of \mathcal{S}_2 satisfies $\mathbb{E}[\rho_0(\mathcal{S}_2)] \leq bs \ln(\frac{s}{s-t}) \text{opt}_{cp}$.*

Proof. Observe that each of the second “for” loop of Algorithm 1 is in fact a realization of Lovász extension in Theorem 2.5 (one may refer to [3]). So the expectation of the cost of those sets in each iteration is $\hat{\rho}(x^*) = \text{opt}_{cp}$. Since \mathcal{S}_2 is the union of these sets, so after $bs \ln(\frac{s}{s-t})$ iterations, $\mathbb{E}[\rho_0(\mathcal{S}_2)] \leq bs \ln(\frac{s}{s-t}) \text{opt}_{cp}$. \square

In the following, when we say that element e is fully covered by \mathcal{S}_2 , it means that the *remaining* covering requirement of e is satisfied by \mathcal{S}_2 . Using such a convention, we denote by $\mathcal{C}(\mathcal{S}_2)$ the set of elements fully covered by \mathcal{S}_2 , and let $p(\mathcal{S}_2) = \sum_{e \in \mathcal{C}(\mathcal{S}_2)} p(e)$. Notice that \mathcal{S}_2 is in fact a random sub-collection, and thus $p(\mathcal{S}_2)$ is a random value. To be more strict, let \hat{y}_e be the random variable which takes value 1 if e is fully covered by \mathcal{S}_2 , and takes value 0 otherwise. Then

$$p(\mathcal{S}_2) = \sum_{e \in \mathcal{C}(\mathcal{S}_2)} p(e) \hat{y}_e. \quad (17)$$

The next lemma gives an upper bound for the expected value of $p(\mathcal{S}_2)$.

Lemma 3.9. *For the collection of sets \mathcal{S}_2 computed by Algorithm 1, the expected profit of \mathcal{S}_2 satisfies $\mathbb{E}[p(\mathcal{S}_2)] \geq tq'P$.*

Proof. Since $\mathbb{E}[p(\mathcal{S}_2)] = \sum_{e \in E'} p(e) \Pr[\hat{y}_e = 1]$ and

$$\sum_{e \in E'} p(e) y_e^* \geq qP - \sum_{e \in \mathcal{C}(\mathcal{S}_1)} p(e) y_e^* \geq qP - \sum_{e \in \mathcal{C}(\mathcal{S}_1)} p(e) = q'P,$$

it suffices to prove that for each $e \in E'$,

$$\Pr[\hat{y}_e = 1] \geq ty_e^*. \quad (18)$$

Notice that for each $e \in E'$, $y_e^* \leq 1/s$. Since we have assumed $t < s$, so $ty_e^* < 1$. Then, proving (18) is equivalent to proving

$$\Pr[\hat{y}_e = 0] \leq 1 - ty_e^*. \quad (19)$$

In a “for” loop with a uniformly randomly chosen $\theta \in [0, 1]$, an r_e -cover \mathcal{A} is chosen into \mathcal{S}_2 if and only if $x_{\mathcal{A}}^* \geq \theta$. For an element $e \in E'$, it is not fully covered by those sets chosen into \mathcal{S}_2 in this “for” loop if and only if $\theta > \max\{x_{\mathcal{A}}^* : \mathcal{A} \in \Omega_e\}$. This occurs with probability $1 - \max_{\mathcal{A} \in \Omega_e} x_{\mathcal{A}}^*$. Since $\max_{\mathcal{A} \in \Omega_e} x_{\mathcal{A}}^* \geq \frac{y_e^*}{b}$ (see (16)), we have

$$1 - \max_{\mathcal{A} \in \Omega_e} x_{\mathcal{A}}^* \leq 1 - \frac{y_e^*}{b}.$$

So, after $bs \ln(\frac{s}{s-t})$ iterations,

$$\Pr[\hat{y}_e = 0] \leq \left(1 - \frac{y_e^*}{b}\right)^{bs \ln(\frac{s}{s-t})} \leq e^{-sy_e^* \ln(\frac{s}{s-t})} = \left(\frac{s}{s-t}\right)^{-sy_e^*},$$

where the second inequality uses the fact that $1 - x \leq e^{-x}$. Denote $f(x) = (\frac{s}{s-t})^{-sx}$ and $g(x) = 1 - tx$. Notice that $f(x)$ is a convex function and $g(x)$ is a linear function. Furthermore, $f(0) = g(0)$, $f(1/s) = g(1/s)$. So $f(x) \leq g(x)$ in interval $[0, 1/s]$. Since for each $e \in E'$, $0 \leq y_e^* \leq 1/s$. So, $(\frac{s}{s-t})^{-sy_e^*} \leq 1 - ty_e^*$. Property (19) is proved, and the lemma follows. \square

Remark 3.10. One may be wondering what if $tq'P$ is larger than the profit of those remaining elements which are not fully covered by \mathcal{S}_1 . This cannot happen because after the first stage of deterministic rounding, the total profits of remaining elements is $q'P + (1 - q)P$. Since it is required that $1 < t < s \leq 1/q \leq 1 + \frac{1-q}{q'}$, we have $tq'P < q'P + (1 - q)P$.

Now we will show that by choosing suitable parameters s and t , Algorithm 1 produces a feasible solution with performance ratio $O(b)$ with high probability.

Theorem 3.11. *Setting $s = 1/q$ and $t = 1/\sqrt{q}$, Algorithm 1 produces a feasible solution to SCPMC with high probability whose cost is $O(b)opt_{cp}$, where $b = \max_e \left(\frac{f}{r_e}\right)$.*

Proof. Notice that for the above s and t , we have $1 < \frac{1}{\sqrt{q}} < t = \frac{1}{\sqrt{q}} < s = \frac{1}{q}$.

The outline of the proof is as follows: we first show that the sum of the probabilities for the following two events is a constant strictly smaller than 1; then a feasible solution with desired performance ratio can be achieved with high probability by repeating Algorithm 1 $O(\ln(n))$ times. The two events are:

- (i) $\rho_0(\mathcal{S}_2) > bsl \ln(\frac{s}{s-t})opt_{cp}$, where $l = \frac{1-q}{(t-1)\varepsilon}$;

(ii) $p(\mathcal{S}_2) < q'P$.

For event (i), using Markov inequality and Lemma 3.8, we have

$$Pr \left[\rho_0(\mathcal{S}_2) > bsl \ln \left(\frac{s}{s-t} \right) opt_{cp} \right] \leq \frac{1}{l} = \frac{(t-1)\varepsilon}{1-q} = 1 - \frac{(1-q) + (1-t)\varepsilon}{1-q}. \quad (20)$$

For event (ii), since $q' > \frac{qP - (q-\varepsilon)P}{P} = \varepsilon$ by Algorithm 1, $q'P + (1-q)P - p(\mathcal{S}_2) \geq 0$ by Remark 3.10, and $E[q'P + (1-q)P - p(\mathcal{S}_2)] \leq q'P + (1-q)P - tq'P$ by Lemma 3.9, using Markov inequality,

$$\begin{aligned} Pr [p(\mathcal{S}_2) \leq q'P] &= Pr [q'P + (1-q)P - p(\mathcal{S}_2) \geq (1-q)P] \\ &\leq \frac{q'P + (1-q)P - tq'P}{(1-q)P} < \frac{(1-q) + (1-t)\varepsilon}{1-q}. \end{aligned} \quad (21)$$

Adding inequalities (20) and (21), the probability that either event (i) occurs or event (ii) occurs is upper bounded by a constant which is strictly smaller than 1. Hence, by repeating Algorithm 1 $O(\ln(n))$ times, with a high probability, $p(\mathcal{S}_2) \geq q'P$ and $\rho_0(\mathcal{S}_2) \leq bsl \ln(\frac{s}{s-t})opt_{cp}$. Combining these with Lemma 3.7, with high probability, $p(\mathcal{S}') = p(\mathcal{S}_1) + p(\mathcal{S}_2) \geq qP$ and

$$\rho_0(\mathcal{S}') \leq \rho_0(\mathcal{S}_1) + \rho_0(\mathcal{S}_2) \leq bs \left(1 + l \ln \left(\frac{s}{s-t} \right) \right) opt_{cp} = O \left(\frac{b}{\varepsilon} \right) opt_{cp},$$

where the first inequality uses (2) and the constant in big O is $\frac{1}{\sqrt{q}-q} \left((\frac{1}{\sqrt{q}-1})\varepsilon + (1-q) \ln(\frac{1}{1-\sqrt{q}}) \right)$. The theorem is proved. \square

4 Approximation Algorithm for SCPSC

As a corollary of Theorem 3.11, the minimum submodular cost partial set cover problem (SCPSC for short, in which the covering requirement for each element is one) admits a bicriteria randomized $(\varepsilon, O(\frac{f}{\varepsilon}))$ -approximation. In the following, we show that an adaptation of our method can yield the same approximation for SCPSC even if the submodular function ρ_0 is non-monotone. The idea behind the adaptation is that in this case, a natural constraint is sufficient (we do not need to use the more complicated r_e -covers), and thus a technique similar to that in [10] dealing with non-monotone submodular functions can be used.

The SCPSC problem can be modelled as the following integer program:

$$\begin{aligned}
& \min \rho_0(x) \\
& s.t. \sum_{e: e \in E} p_e y_e \geq qP, \\
& \sum_{S: S \in \mathcal{S}} x_S \geq y_e, \text{ for any } e \in E \\
& x_S \in \{0, 1\} \text{ for } S \in \mathcal{S} \\
& y_e \in \{0, 1\} \text{ for } e \in E,
\end{aligned}$$

Its relaxation is a convex program:

$$\begin{aligned}
& \min \hat{\rho}_0(x) \\
& s.t. \sum_{e: e \in E} p_e y_e \geq qP, \\
& \sum_{S: S \in \mathcal{S}} x_S \geq y_e, \text{ for any } e \in E \\
& x_S \geq 0 \text{ for } S \in \mathcal{S} \\
& 1 \geq y_e \geq 0 \text{ for } e \in E
\end{aligned} \tag{22}$$

Notice that since we can use ρ_0 as objective function here, the convexity follows directly from the submodularity of ρ_0 . While for program (11), its convexity is guaranteed by Lemma 3.4, which is no longer true if ρ_0 is non-monotone (see Remark 3.5).

Define a new function γ by $\gamma(\mathcal{S}') = \min\{\rho_0(\mathcal{S}'') : \mathcal{S}' \subseteq \mathcal{S}'' \subseteq \mathcal{S}\}$. Then γ is a nonnegative monotone nondecreasing submodular function (see [10]). For any sub-collection $\mathcal{S}' \subseteq \mathcal{S}$, the value of $\gamma(\mathcal{S}')$ can be determined in polynomial time by an algorithm for submodular function minimization. Let \mathcal{S}'_0 be the minimizer, that is, $\mathcal{S}' \subseteq \mathcal{S}'_0 \subseteq \mathcal{S}$ and $\rho(\mathcal{S}'_0) = \gamma(\mathcal{S}')$. It should be noticed that \mathcal{S}'_0 can fully cover all those elements which are fully covered by \mathcal{S}' (since $\mathcal{S}' \subseteq \mathcal{S}'_0$).

Our algorithm for SCPSC is similar to Algorithm 1 with the following two differences. First, replace convex program (11) by (22). Second, having obtained \mathcal{S}_1 , compute $(\mathcal{S}_1)_0$ and replace \mathcal{S}_1 by $(\mathcal{S}_1)_0$ in the remaining part of Algorithm 1.

Notice that in the analysis, monotonicity is used only in Lemma 3.7. So, to obtain the desired result, we only need to prove the following lemma.

Lemma 4.1. $\rho((\mathcal{S}_1)_0) \leq bs \cdot \text{opt}_{cp}$.

Proof. Let \hat{x} be the indicator vector of \mathcal{S}_1 . By the monotonicity of γ , the Lovász extension $\hat{\gamma}$ is also monotone nondecreasing. Hence it follows from $\hat{x} \leq bsx^*$ that

$$\hat{\gamma}(\hat{x}) \leq \hat{\gamma}(bsx^*). \tag{23}$$

For any sub-collection $\mathcal{S}' \subseteq \mathcal{S}$, by Definition 2.3, $\gamma(\mathcal{S}') \leq \rho(\mathcal{S}')$. By the definition of Lovász extension in Definition 2.3, we have

$$\hat{\gamma}(x) \leq \hat{\rho}(x) \text{ holds for any vector } x \in [0, 1]^{|\mathcal{S}|}. \tag{24}$$

Combining (23), (24) with the positive homogeneous property of Lovász extension,

$$\rho((\mathcal{S}_1)_0) = \gamma(\mathcal{S}_1) = \hat{\gamma}(\hat{x}) \leq \hat{\gamma}(bsx^*) = bs\hat{\gamma}(x^*) \leq bs\hat{\rho}(x^*) = bs \cdot opt_{cp}.$$

The lemma is proved. \square

From the above argument, we have the following result.

Theorem 4.2. *For any nonnegative submodular function, the SCPSC problem has a bicriteria randomized $(\varepsilon, O(\frac{f}{\varepsilon}))$ -approximation with high probability.*

5 Conclusion

By introducing a novel convex program describing the minimum submodular cost partial multi-cover problem (SCPMC), we give a randomized $(\varepsilon, O(\frac{b}{\varepsilon}))$ -approximation algorithm for SCPMC, where $b = \max_e \binom{f}{r_e}$. Since PMC is a special case of SCPMC, the PMC problem also has a bicriteria randomized $(\varepsilon, O(\frac{b}{\varepsilon}))$ -approximation algorithm with a high probability. We show that in the case when the covering requirement for each element is one, monotonicity requirement can be dropped off from the cost function. It should be noticed that if we only care about an expected result, then we may obtain a randomized algorithm producing a sub-collection \mathcal{S}' with $\mathbb{E}[\rho_0(\mathcal{S}')] \leq bs(1 + \ln \frac{s}{s-t})opt$ and $\mathbb{E}[p(\mathcal{S}')] \geq qP$. This can be achieved by modifying $(q - \varepsilon)P$ in Line 8 of Algorithm 1 into qP .

One question is can one obtain the same result for SCPMC without monotonicity requirement? Another question is what if r_{\max} is not upper bounded by a constant?

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